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Yagita invariant of mapping class groups at the prime 2

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Abstract

We compute the Yagita invariant of mapping class groups at the prime number 2.

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0. Introduction

The mapping class group Γ_g is defined to be the group of path components of the group of orientation-preserving diffeomorphisms of an oriented closed surface S_g of genus g.

Let Γ be a group of finite virtual cohomological dimension and $\pi \subset \Gamma$ any subgroup of prime order p. There exists a maximum value $m = m(\pi)$ such that

$$\operatorname{Im}((H^*(\Gamma;\mathbb{Z}) \to H^*(\pi;\mathbb{Z}/p)) \subset \mathbb{Z}/p[u^m]) \subset H^*(\pi;\mathbb{Z}/p),$$

where the map is the restriction map and $u \in H^2(\pi;\mathbb{Z}/p)$ is a generator. Recall that the Yagita invariant $p(\Gamma)$ of Γ with respect to the prime p is then defined to be the least common multiple of values $2m(\pi)$, where π ranges over all subgroups of order p of Γ (see [9, 7, 4]). Notice that $p(\Gamma)$ is defined to be 1 if Γ does not contain any subgroup of order p.

The Yagita invariant $p(\Gamma)$ generalizes the p-period of a group. As it is the case for the p-period, $p(\Gamma)$ divides $2(p-1)p^k$, for some $k \geq 0$. Especially, the invariant $2(\Gamma)$ is of the form 2^k for some $k \geq 0$. The invariant $p(\Gamma_g)$ is calculated for an odd regular prime by Glover, Mislin and the author in [3, 4]. For even genus $2(\Gamma_{2h})$ is obtained by the author in [8]. In this note, we will calculate $2(\Gamma_g)$ in some interesting special cases.

Our main result is stated as follows.

Theorem 1. *Assume that* $g = l2^{2^k-1} + 1$ *with* $l \geq 2^{k+1} - 1$ *an odd integer and* $k \geq 0$. *Then,* $2(\Gamma_g)$ *is* 2^{2^k+1}.

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When $k = 0$, Theorem 1 is the main result in [8] which states that $2(\Gamma_g)$ is 4 if g is even. When $k = 1$, Theorem 1 says $2(\Gamma_g)$ is 8 if $g > 3$ is $3 \bmod (4)$. A direct observation that Γ_3 contains the quaterion group of order 8 (or see [1]) and Theorem 2 below in the case $\alpha = 1$ imply that the invariant $2(\Gamma_3)$ is 4. Theorem 1 suggests a conjecture that the invariant $2(\Gamma_g)$ is $2^{\alpha+2}$ if $g = l2^{\alpha} + 1$ with l odd and sufficiently large. As a complement of Theorem 1, we also prove

Theorem 2. *Assume that $g = 2^{\alpha} + 1$ with $\alpha > 0$. Then, the Yagita invariant $2(\Gamma_g)$ is either 2^{α} or $2^{\alpha+1}$.*

Theorem 3. *Assume that $g = l2^{\alpha} + 1$ with l an odd integer and $\alpha \geq 0$. Then, the Yagita invariant $2(\Gamma_g)$ is either 2^{α}, $2^{\alpha+1}$ or $2^{\alpha+2}$.*

There are two main techniques in this note different from previous approaches for calculating the Yagita invariant of mapping class groups. The one is that we employ Stiefel–Whitney classes instead of Chern classes to make a more precise upper bound of the invariant $2(\Gamma_g)$. The other one is that we study an elementary abelian 2-group action on the surface of genus g with a certain property to raise the lower bound of the invariant $2(\Gamma_g)$.

The rest of this note is organized as follows. In Section 1, we provide an upper bound for $2(\Gamma_g)$. In Section 2, we get a lower bound for $2(\Gamma_g)$ under the assumptions of Theorem 1. This lower bound agrees with the upper bound in Section 1 if $g = l2^{2^k-1} + 1$ ($l \geq 2^{k+1} - 1$ an odd number). Thus, Theorem 1 is proved in this section. In Section 3, we obtain a sharp upper bound for $2(\Gamma_g)$ in the case $g = 2^{\alpha} + 1$ and finish the proof of Theorem 2. Theorem 3 follows by combining Proposition 1.3 and Lemma 3.3.

1. An upper bound for $2(\Gamma_g)$

Let

$$\rho : \Gamma_g \to Sp_{2g}(\mathbb{R}) \to Gl_{2g}(\mathbb{R})$$

denote the homology representation by letting Γ_g act on $H_1(S_g;\ \mathbb{R})$. The Stiefel–Whitney class $w_i(\rho) \in H^i(\Gamma_g; \mathbb{Z}/2)$ is defined via the flat $\mathbb{R}^{2g}$ bundle classified by the map

$$B\rho : K(\Gamma_g, 1) \to BGl_{2g}(\mathbb{R}).$$

Let $U(g)$ be a maximal compact subgroup of $Sp_{2g}(\mathbb{R})$. It is well known that $BSp_{2g}(\mathbb{R})$ is homotopy equivalent to $BU(g)$; thus,

$$H^*(BSp_{2g}(\mathbb{R}); \mathbb{Z}/2) = \mathbb{Z}/2[d_1, d_2, \ldots, d_g]$$

($deg(d_i) = 2i$, $i \geq 1$), where the d_i is exactly corresponding to the universal Chern class $c_i \in H^{2i}(BU(g);\ \mathbb{Z}/2)$. These d_i's are the reductions of some cohomology elements in $H^*(BSp_{2g}(\mathbb{R});\mathbb{Z})$ with integer coefficients since Chern classes c_i's are defined in $H^*(BU(g);\mathbb{Z})$ with integer coefficients. Let $i : Sp_n(\mathbb{R}) \to Gl_n(\mathbb{R})$ be the natural inclusion. Notice that a general relation $i^*(w_{2i}) = d_i$ implies that

$$w_{2i}(\rho) \in H^{2i}(\Gamma_g;\mathbb{Z}/2)$$

is a reduction of a cohomology element in $H^{2i}(\Gamma_g;\mathbb{Z})$.

Assume that $g = l2^\alpha + 1$ with $\alpha \geq 0$ and l an odd integer. Let $\pi = \mathbb{Z}/2 \subset \Gamma_g$ be a subgroup. One may think of π as a $\mathbb{Z}/2$ action on S_g associated with the Riemann–Hurwitz equation

$$2g - 2 = 2(2h - 2) + n,$$

where $h \geq 0$ is the genus of the orbit space $S_g/\mathbb{Z}/2$ and $n \neq 1$ is the number of fixed points of the $\mathbb{Z}/2$ action on S_g. Let $\rho_\pi : \pi \to \Gamma_g$ denote the representation of the restriction of ρ to $\pi \subset \Gamma_g$, and the Stiefel–Whitney class $w_i(\rho_\pi)$ denote the restriction of $w_i(\rho)$ under ρ_π.

Lemma 1.1. $w_{2(g-h)}(\rho_\pi) \neq 0$.

Proof. Let ρ_0 denote the trivial representation and ρ_1 denote the unique irreducible representation of $\mathbb{Z}/2 = \langle t \rangle$. Then one obtains easily

$$\rho_\pi = 2(g-h)\rho_1 \oplus 2h\rho_0,$$

by combining the Lefschetz formula $Tr(\rho_\pi(t)) = 2 - n$ and the Riemann–Hurwitz formula $2g-2 = 2(2h-2)+n$. Thus, the total Stiefel–Whitney class $w(\rho_\pi) = (1+x^2)^{g-h}$ implies that the Stiefel–Whitney class $w_{2(g-h)}(\rho_\pi) \neq 0$ in $H^{2(g-h)}(\pi;\mathbb{Z}/2)$. □

A similar argument as in Section 4 of [4] implies the following lemma which is similar to Proposition 4.3 in [4] there stated for p an odd prime.

Lemma 1.2. *Let $g > 2$ and $\pi \subset \Gamma_g$ be a subgroup of order 2, with the associated Riemann–Hurwitz formula $2g - 2 = 2(2h - 2) + n$. Then there exists a cohomology element $e \in H^{6(g-h)-2n}(\Gamma_g;\mathbb{Z})$ whose restriction to $H^{6(g-h)-2n}(\pi;\mathbb{Z})$ is nontrivial.*

Combining the two types of cohomology elements in Lemmas 1.1 and 1.2 above together, we give a general upper bound of the invariant $2(\Gamma_g)$.

Proposition 1.3. *Assume $g = l2^\alpha + 1$ with l odd and $\alpha \geq 0$. Then the invariant $2(\Gamma_g)$ divides $2^{\alpha+2}$. In particular, if g is even, then $2(\Gamma_g)$ divides 4.*

Proof. For every $\pi \subset \Gamma_g$ of order 2, we need to find a cohomology element

$$e \in H^{2^{\beta(\pi)} j(\pi)}(\Gamma_g;\mathbb{Z})$$

($\beta(\pi) \leq \alpha + 2$, $j(\pi)$ odd) so that the restriction of e to $H^{2^{\beta(\pi)} j(\pi)}(\pi;\mathbb{Z})$ is nontrivial.

Case 1: $2g-2+n \not\equiv 0 \bmod (2^{\alpha+4})$. It is easy to verify that $2(g-h)=(2g-2+n)/2 \not\equiv 0 \bmod (2^{\alpha+3})$ because of $h=(2g+2-n)/4$. Thus, we take $e \in H^{2^{\beta(\pi)}j(\pi)}(\Gamma_g;\mathbb{Z})$ as a lift of $w_{2(g-h)}(\rho) \in H^{2^{\beta(\pi)}j(\pi)}(\Gamma_g;\mathbb{Z}/2)$.
Case 2: $2g-2+n \equiv 0 \bmod (2^{\alpha+4})$. Notice that $6(g-h)-2n = 1/2(6g-6-n)$. We claim that $6g-6-n \not\equiv 0 \bmod (2^{\alpha+4})$. Then, we take e to be the cohomology element given in Lemma 1.2. In fact, if both $2g-2+n$ and $6g-6-n$ are $0 \bmod (2^{\alpha+4})$, then $8g-8 \equiv 0 \bmod (2^{\alpha+4})$, i.e., $g \equiv 1 \bmod (2^{\alpha+1})$. This contradicts our assumption. □

2. A lower bound for $\lambda(\Gamma_g)$

In this section, we assume $g = l2^{2^k-1}+1$ ($l \geq 2^{k+1}-1$ odd and $\alpha \geq 0$). The case $g = 2^\alpha + 1$ will be treated in Section 3. If there is a finite group G acting on S_g one may consider this action as a subgroup $G \subset \Gamma_g$. The idea of this section is to construct an elementary abelian 2-group $E = \langle a_1,\dots,a_{2^k}\rangle$ of rank 2^k acting on S_g so that, for any a_i,a_j, $1 \leq i < j \leq 2^k$, there is an element $n_{i,j} \in \Gamma_g$ satisfying $n_{i,j}a_in_{i,j}^{-1} = a_j$, $n_{i,j}a_jn_{i,j}^{-1} = a_i$ and $n_{i,j}a_kn_{i,j}^{-1} = a_k$ for $k \neq i,j$. We abuse the notation E here. Then, we prove that the invariant $\lambda(N_{\Gamma_g}(E))$ (the normalizer of E) is a multiple of 2^{2^k+1}, so is the invariant $\lambda(\Gamma_g)$.

Proposition 2.1. *Assume $g = l2^{2^k-1}+1$ ($l \geq 2^{k+1}-1$ odd, and $k \geq 1$). There is an elementary abelian 2-group $E = \langle a_1,a_2,\dots,a_{2^k}\rangle$ of rank 2^k acting on S_g so that, for any a_i,a_j, $1 \leq i < j \leq 2^k$, there is an element $n_{i,j}$ in Γ_g satisfying $n_{i,j}a_in_{i,j}^{-1} = a_j$, $n_{i,j}a_jn_{i,j}^{-1} = a_i$ and $n_{i,j}a_kn_{i,j}^{-1} = a_k$ for $k \neq i,j$.*

Proof. We construct a surjective map

$$\mu : \pi_1(S_{(l+1)/2} - \{x_1,x_2\}) \to E = \langle a_0,a_1,\dots,a_{2^k}\rangle$$

with $\mu(b_0)=a_0$, $\mu(b_1)=a_1,\dots,\mu(b_{2^k})=a_{2^k}, \mu(b_i)=1$ for $2^k \leq i \leq (l+1)/2$ and $\mu(c_i)=1$ for $0 \leq i \leq (l+1)/2$, and $\mu(x_1)=\mu(x_2)=a_0a_1\cdots a_{2^k}$, where b_i, c_i and x_j for $0 \leq i \leq \alpha$ and $1 \leq j \leq 2$ consist of a set of generators of $\pi_1(S_{(l+1)/2} - \{x_1,x_2\})$. Note that $(l+1)/2 \geq 2^k$ by assumption. This surjection μ gives rise to a covering map

$$p : S_g \to S_{(l+1)/2}$$

with two branch points with the deck transformation E of S_g since the Riemann–Hurwitz formula

$$2g-2 = l2^{2^k} = 2^{2^k}(2h-2) + 2^{2^k}(1-\tfrac{1}{2})n$$

holds when taking $h=(l+1)/2$ and $n=2$. Consider an automorphism $\beta_{i,j}$ of E defined by $\beta_{i,j}(a_i)=a_j$, $\beta_{i,j}(a_j)=a_i$ and $\beta_{i,j}(a_k)=a_k$ for $k \neq i,j$. Notice that the map

μ factors through the homology and there is a surjective map

$$\bar{\mu} : H_1(S_{(l+1)/2} - \{x_1, x_2\}; \mathbb{Z}) \to E$$

with $\mu = \bar{\mu}\pi$, where π is the abelianization map from $\pi_1(S_{(l+1)/2} - \{x_1, x_2\})$ to $H_1(S_{(l+1)/2} - \{x_1, x_2\}; \mathbb{Z})$. We use again b_i, c_i and x_1 as elements in the basis of $H_1(S_{(l+1)/2} - \{x_1, x_2\}; \mathbb{Z})$. We may also assume that b_i, c_i are symplectic. Then, the $(l+2) \times (l+2)$ matrix

$$X = \begin{pmatrix} E_{i,j} & 0 & 0 \\ 0 & (E_{i,j}^T)^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

gives rise to an automorphism $\gamma_{i,j}$ of $H_1(S_{(l+1)/2} - \{x_1, x_2\}; \mathbb{Z})$ which preserves the symplectic form such that $\bar{\mu}\gamma_{i,j} = \beta_{i,j}\bar{\mu}$, where $E_{i,j}$ is the $(l+1)/2 \times (l+1)/2$ matrix which exchanges ith and jth rows of the identity matrix. Such $\gamma_{i,j} \in Aut\,(H_1(S_{(l+1)/2} - \{x_1, x_2\}; \mathbb{Z}))$ can be realized by a homeomorphism $f_{i,j}$ of $S_{(l+1)/2}$ which fixes points x_1 and x_2 since the natural map from Γ_g to the symplectic group $Sp(2g, \mathbb{Z})$ is surjective. So, there is a homeomorphism $n_{i,j}$ of S_g which lifts $f_{i,j}$ in the sense $pf_{i,j} = n_{i,j}p$ by a classical result of MacLachlan and Harvey (see [5]). Recall that the map p is the branched covering map from S_g to $S_{(l+1)/2}$ with two branch points. This homeomorphism $n_{i,j}$ is the one we need in this proposition. □

Lemma 2.2. *Let Γ be a group of finite vcd and $E \subset \Gamma$ an elementary abelian 2-subgroup of rank 2^k ($k > 0$). Assume that a basis of E is $\langle a_1, a_2, \ldots a_{2^k}\rangle$. For any a_i, a_j ($1 \le i < j \le 2^k$), if there is an element $n_{i,j} \in \Gamma$ such that $n_{i,j}a_in_{i,j}^{-1} = a_j$, $n_{i,j}a_jn_{i,j}^{-1} = a_i$ and $n_{i,j}a_mn_{i,j}^{-1} = a_m$ for $1 \le m \le 2^k$ and $m \ne i,j$, then the Yagita invariant $\mathfrak{Z}(N(E))$ is a multiple of 2^{2^k+1}. So is $\mathfrak{Z}(\Gamma)$.*

Proof. We show that the diagonal embedding

$$\Delta : \mathbb{Z}/2 \to E \subset N(E)$$

induces a restriction map

$$\Delta^* : H^*(N(E); \mathbb{Z}) \to H^*(\mathbb{Z}/2; \mathbb{Z})$$

mapping into $\mathbb{Z}[w^{2^{2^k}}]/2w^{2^{2^k}}$, where $w \in H^2(\mathbb{Z}/2; \mathbb{Z})$ is a generator. Consider the diagonal restriction map

$$\rho^* : H^*(E; \mathbb{Z}) \to H^*(\mathbb{Z}/2; \mathbb{Z}).$$

Note that it is well known that

$$H^*(E; \mathbb{Z}) = \mathbb{Z}[w_1, \ldots, w_{2^k}]/2(w_1, \ldots, w_{2^k}),$$

where each w_i of degree 2 and $(w_1, \ldots, w_{2^k})$ denoting the ideal generated by these elements. All elementary symmetric functions in the variables $w_1, \ldots, w_{2^k}$ map via

ρ^* to 0 so that the image of Δ^* is contained in the subalgebra generated by the image of $\prod w_i$, which is w^{2^k}. This implies that $2(N(E))$ is a multiple of 2^{2^k+1}. Thus, $2(\Gamma)$ is a multiple of 2^{2^k+1}. □

The combination of Proposition 2.1 and Lemma 2.2 gives

Corollary 2.3. *Assume that $g = l2^{2^k-1} + 1$ with $l \geq 2^{k+1} - 1$ odd and $k \geq 0$. Then the Yagita invariant $2(\Gamma_g)$ is a multiple of 2^{2^k+1}.*

Proposition 1.3, Corollary 2.3 and the result in the case $k = 0$ given in [8] together imply Theorem 1 in introduction.

3. Yagita invariant $2(\Gamma_{2^\alpha+1})$

We provide a sharp upper bound of the invariant $2(\Gamma_g)$ in the case $g = 2^\alpha + 1$. Let π denote a cyclic action of order 2 on S_g with the associated Riemann–Hurwitz formula

$$2g - 2 = 2(2h - 2) + n,$$

where h is the genus of the orbit space and the n is the number of fixed points of the π action.

Lemma 3.1. *If $g = 2^\alpha + 1$ with $\alpha > 0$, then $2(g - h) \not\equiv 0 \bmod (2^{\alpha+2})$.*

Proof. Observe that $h = (2g + 2 - n)/4$. Thus,

$$0 \leq n \leq 2g + 2 = 2^{\alpha+1} + 4.$$

Then, one obtains that $2(g-h) = (2g-2+n)/2 \not\equiv 0 \bmod (2^{\alpha+2})$. Otherwise, $2g-2+n = 2^{\alpha+1} + n \equiv 0 \bmod (2^{\alpha+3})$ implies that

$$n \geq 2^{\alpha+3} - 2^{\alpha+1} > 2^{\alpha+1} + 4. \qquad \square$$

The combination of Lemmas 1.1 and 3.1 gives a cohomology element

$$e \in H^{2^{\beta(\pi)}j(\pi)}(\Gamma_g; \mathbb{Z})$$

such that $Res|_\pi(e) \neq 0$ for $\beta(\pi) \leq \alpha + 1$. So we obtain

Lemma 3.2. *Let $g = 2^\alpha + 1$ with $\alpha > 0$. Then, the invariant $2(\Gamma_g)$ divides $2^{\alpha+1}$.*

Assume that $g = l2^\alpha + 1$ with l any odd integer and $\alpha \geq 0$. It is straightforward to find a cyclic subgroup H of order $2^{\alpha+1}$ in Γ_g so that the index of $[N(H)\colon C(H)]$ is 2^α. Such H could be realized by constructing a cyclic action of order $2^{\alpha+1}$ on S_g with 2 singular orbits and the quotient space a surface $S_{(l+1)/2}$ of genus $(l + 1)/2$. The associated Riemann–Hurwitz equation is

$$2g - 2 = 2^{\alpha+1}(2h - 2) + 2^{\alpha+1}(1 - \tfrac{1}{2})2.$$

The cyclic $2^{\alpha+1}$ fold covering

$$p : S_g - \{2pts\} \to S_{(l+1)/2} - \{2pts\}$$

is given by a surjective map

$$f : \pi_1(S_{1/2(l+1)} - \{2pts\}) = \langle a, b, x_1, x_2 \mid [ab]x_1x_2 = 1 \rangle \to H = \langle y \rangle$$

such that $f(a_1) = y$, $f(a_i) = 1$ for $2 \le i \le \frac{1}{2}(l+1)$, $f(b_i) = 1$ for $1 \le i \le \frac{1}{2}(l+1)$ and $f(x_1) = f(x_2) = y^{2^{\alpha}}$. The deck transformation group associated to p is denoted by H. One can see that the index of $[N(H)\colon C(H)]$ is 2^{α} from the fixed-point data (see [6]). This implies that the Yagita invariant $2(N(H))$ is a multiple of 2^{α}. Namely, we have

Lemma 3.3. *Let $g = l2^{\alpha} + 1$ with l odd and $\alpha \ge 0$. Then, the invariant $2(\Gamma_g)$ is a multiple of 2^{α}.*

Theorem 2 in the introduction follows from Lemmas 3.2 and 3.3, and Theorem 3 in the introduction follows from Proposition 1.3 and Lemma 3.3.

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